

LARGE DEVIATION FOR OUTLYING COORDINATES IN β ENSEMBLES

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ABSTRACT. For Y a subset of the complex plane, a β ensemble is a sequence of probability measures $Prob_{n,\beta,Q}$ on Y^n for $n = 1, 2, \dots$ depending on a positive real parameter β and a real-valued continuous function Q on Y . We consider the associated sequence of probability measures on Y where the probability of a subset W of Y is given by the probability that at least one coordinate of Y^n belongs to W . With appropriate restrictions on Y, Q we prove a large deviation principle for this sequence of probability measures. This extends a result of Borot-Guionnet to subsets of the complex plane and to β ensembles defined with measures using a Bernstein-Markov condition.

1. INTRODUCTION

β ensembles are generalizations of the joint probability distributions of the classical matrix ensembles. A large deviation principle (l.d.p.) for the largest eigenvalue in a β ensemble on the real line was one of the results established in [7]. This question had been considered in ([1], section 2.6.2) where it was established using an assumption ([1], assumption 2.6.5). The result of [7] showed the assumption to hold for β ensembles on the real line.

The main result of this paper is to extend that result to β ensembles on subsets Y of the complex plane. Of course there is no "largest" coordinate of a point in \mathbb{C}^n . However, at least insofar as the rate function for a l.d.p. is concerned, it is equivalent to consider the probability that at least one coordinate or, if fact, one particular coordinate, say

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z_1 belongs to a subset of Y .

This is because the joint probability distribution is symmetric in z_1, \dots, z_n so

$$n \text{Prob}\{z_1 \in W\} \geq \text{Prob}\{\text{at least one of } z_1, \dots, z_n \in W\} \geq \text{Prob}\{z_1 \in W\},$$

where W is a subset of Y . Furthermore, the l.d.p. has speed n . This latter form, i.e. in terms of the coordinate z_1 is how we will specifically develop the l.d.p.

We now consider precise definitions. We will consider the family of probability distributions $\text{Prob}_{n,\beta,Q}$ for $n = 1, 2, \dots$ defined on a closed regular (in the sense of potential theory) subset $Y \subset \mathbb{C}$ given as follows: Let $\beta > 0$ and let Q be a continuous real-valued function on Y . We assume that $R = 2Q/\beta$ is of superlogarithmic (see 6.1) growth if Y is unbounded. Now $\text{Prob}_{n,\beta,Q}$ is defined on Y^n by

$$(1.1) \quad \text{Prob}_{n,\beta,Q} = \frac{A_{n,\beta,Q}(z)}{Z_{n,\beta,Q}} d\tau(z)$$

where

$$(1.2) \quad A_{n,\beta,Q}(z) := |VDM(z_1, \dots, z_n)|^\beta \exp(-2n[Q(z_1) + \dots + Q(z_n)]),$$

VDM denotes the VanDerMonde determinant, and the normalizing constants $Z_{n,\beta,Q}$ are given by:

$$(1.3) \quad Z_{n,\beta,Q}(Y) = Z_{n,\beta,Q} := \int_{Y^n} A_{n,\beta,Q}(z) d\tau(z).$$

Here

$$d\tau(z) = d\tau(z_1) \dots d\tau(z_n)$$

and τ is an appropriate measure on Y . (The existence of the integrals in the case of Y unbounded is dealt with in section 6).

We refer to the family of such probability measures as a β ensemble.

For $Y = \mathbb{R}$, $d\tau = dx$ (Lebesgue measure), $Q(x) = x^2/2$ and $\beta = 1, 2, 4$ we obtain the joint probability distribution of the eigenvalues of the classical matrix ensembles-respectively the Gaussian orthogonal, unitary and symplectic ensembles [1]. In this context, the coordinates of a point $x = (x_1, \dots, x_n)$ are referred to as eigenvalues.

The 2-dimensional version of these probability distributions occurs in the study of the Coulomb gas model ([9],[10]). In this model the parameter β corresponds to the inverse temperature, $2Q$ to the confining potential, and the coordinates of a point are the positions of particles.

These ensembles have been extensively studied (see [1],[9] and the references given there). In particular, it is known that the normalized counting measure of a random point in such ensembles converges weak*, almost surely, to the weighted equilibrium measure. Furthermore a large deviation principle for the normalized counting measures of a random point is known [1].

In this paper we will establish a large deviation principle for the coordinate $z_1 \in Y$ under the probability distributions 1.1. That is we will prove a large deviation principle for the countable family of probability distributions on Y given by

$$(1.4) \quad \psi_n(z_1) = \left(\frac{1}{Z_{n,\beta,Q}} \int_{Y^{n-1}} A_{n,\beta,Q}(z) d\tau(z_2), \dots, d\tau(z_n) \right) d\tau(z_1).$$

for $n = 1, 2, \dots$

The large deviation principle has speed n and rate function

$$(1.5) \quad \mathbb{J}_{Y,\beta,Q}(z_1) = 2Q(z_1) - \beta \left(\int_Y \log |z_1 - t| d\mu_{Y,\beta,Q} + \rho \right)$$

where $\mu_{Y,\beta,Q}$ is an equilibrium measure (see 3.5) and ρ is a constant. We may also express the rate function as

$$(1.6) \quad \mathbb{J}_{Y,\beta,Q}(z_1) = \beta(R(z_1) - V_{Y,R}(z_1)) = 2Q(z_1) - \beta V_{Y,R}(z_1)$$

where $V_{Y,R}$ is the weighted Green function of Y with respect to R (see 2.2) and $R(z_1) = \frac{2}{\beta}Q(z_1)$.

We will assume that Y is regular, and, as a consequence, the rate function will be continuous.

We will also make an assumption (see 2.5) equivalent to assuming that the rate function is strictly positive outside the support of the equilibrium measure.

A key step is theorem 4.1 where we prove a result on the asymptotics of normalizing constants, proved in [7] for subsets of \mathbb{R} and used previously as an assumption in [1]. Our methods use polynomial estimates and potential theory but not the large deviation principle for the normalized counting measures of a random point. We will specifically use weighted potential theory (see [15]), since the probability distributions given by 1.1 are, in each variable a power of the absolute value of a weighted polynomial (see section 2).

We first prove the large deviation result for the coordinate z_1 in the case Y is compact (theorem 5.1) and then in the general case (theorem 6.3). This extends the result of [7] to regular subsets of the plane and, in addition, the measure τ used to define the ensemble can be more general than Lebesgue measure: it need only satisfy the Bernstein-Markov condition. The rate function is independent of τ as long as τ satisfies a Bernstein-Markov condition.

We let C, c denote positive constants which may vary from line to line.

2. POLYNOMIAL ESTIMATES

We will list some basic results of weighted potential theory (see [15]).

Let Y be a regular (in the sense of potential theory), closed set in the plane and R be a continuous (real-valued) function on Y . If Y is unbounded, R is assumed to satisfy 6.1, i.e. to be super-logarithmic.

The weighted equilibrium measure is denoted $\mu_{Y,R}$. It has compact support and is the unique minimizer of

$$(2.1) \quad E(\nu) = - \int \int \log |z - t| d\nu(z) d\nu(t) + 2 \int R(z) d\nu(z)$$

over all measures $\nu \in \mathcal{M}(Y)$ where $\mathcal{M}(Y)$ denotes the probability measures on Y ([15], I Theorem 1.3).

The weighted Green function of Y with respect to R ([15], Appendix B) is denoted by $V_{Y,R}$. It is defined by

$$(2.2) \quad V_{Y,R}(z) = \sup \{u(z) | u \text{ is subharmonic on } \mathbb{C}, u \leq R \text{ on } Y \text{ and, } u \leq \log^+ |z| + C\}.$$

$V_{Y,R}$ is continuous ([14], prop 2.16) since Y was assumed to be regular and any regular compact set in the plane is locally regular.

Now, by ([15] Appendix B, Lemma 2.4)

$$(2.3) \quad V_{Y,R}(z) = \int_Y \log |z - t| d\mu_{Y,R}(t) + \rho.$$

where ρ is the Robin constant given by

$$(2.4) \quad \rho = \lim_{|z| \rightarrow \infty} (V_{Y,R}(z) - \log |z|).$$

It is also known that $E(\mu_{Y,R}) > -\infty$ and ([15], I theorem 3(d))

$$\rho = E(\mu_{Y,R}) - \int_Y R(z) \mu_{Y,R}.$$

We let $S_R^* := \{z \in Y | V_{Y,R}(z) = R(z)\}$ and $S_R = \text{supp}(\mu_{Y,R})$. In fact S_R and S_R^* depend on Y but our notation does not explicitly indicate this. S_R is non-polar and non-polar in a neighbourhood of each of its points.

In general $S_R \subset S_R^*$ but we will make the assumption :

$$(2.5) \quad S_R = S_R^*$$

In [7] the corresponding assumption is referred to as control of large deviations.

From 2.3 we have ([15], I Theorem 3(f))

$$(2.6) \quad R(z) = \int_Y \log |z - t| d\mu_{Y,R}(t) + \rho$$

for $z \in S_R$.

Assumption 2.5 is equivalent to $V_{Y,R}(z) < R(z)$ for $z \notin S_R$ or to assuming that the rate function (see 1.5 or 1.6) $\mathbb{J}_{Y,\beta,Q}(z) > 0$ for $z \notin S_R$. Assumption 2.5 will be used in proving the lower bound for the l.d.p.

We let \mathcal{P}_n denote the polynomials in the single variable z of degree $\leq n$ and for $p \in \mathcal{P}_n$ we refer to $e^{-nR(z)}p(z)$ as a *weighted polynomial of degree n* (the weight is the positive continuous function $e^{-R(z)}$).

Now we set

$$(2.7) \quad R(z) = \frac{2}{\beta} Q(z).$$

Note that $A_{n,\beta,Q}(z)$ is, in each variable of the form $|e^{-nR(z)}p(z)|^\beta$ i.e. the absolute value of a weighted polynomial to the β power. For this reason weighted potential theory can be used in the study of β ensembles.

Equations 2.8 and 2.9 below are known estimates for weighted polynomials.

By ([15], III Corollary 2.6) the sup norm of a weighted polynomial is assumed on S_R . That is

$$(2.8) \quad \|e^{-nR}p(z)\|_Y = \|e^{-nR(z)}p(z)\|_{S_R}.$$

for all $p \in \mathcal{P}_n$.

By ([15], I Theorem 3.6) we have, for p a monic polynomial of degree n

$$(2.9) \quad \|e^{-nR(z)}p(z)\|_{S_R} \geq e^{-n\rho}.$$

Then from 2.9, for p monic of degree n

$$(2.10) \quad \|e^{-2nQ}p^\beta\|_{S_R} \geq e^{-n\rho\beta}.$$

For G a subset of $\mathcal{M}(Y)$ we will use the notation

$$\tilde{G}_n = \{z = (z_1, \dots, z_n) \in Y^n \mid \frac{1}{n} \sum_{j=1}^n \delta(z_j) \in G\}$$

Here δ denotes the Dirac delta function.

We now restrict to the case that Y is compact and we will use the notation K in place of Y .

Lemma 2.1. *Given $\epsilon > 0$, there is a neighborhood G of $\mu_{K,R}$ in $\mathcal{M}(K)$ (with the weak* topology) such that for $(z_1, \dots, z_n) \in \tilde{G}_n$ we have*

$$\|e^{-nR(t)}(t - z_1), \dots, (t - z_n)\|_K \leq e^{-n(\rho - \epsilon)}.$$

Proof. The proof will be by contradiction. If not, for some $\epsilon > 0$, no such G exists. Thus there exists a sequence of n_s -tuples $(z_1^s, \dots, z_{n_s}^s)$ for $s = 1, 2, \dots$ with

$$(2.11) \quad \lim_s \frac{1}{n_s} \sum_{j=1}^{n_s} \delta(z_j^s) = \mu_{K,R}$$

weak*, but, using 2.8,

$$(2.12) \quad \|e^{-n_s R(t)} (t - z_1^s) \dots (t - z_{n_s}^s)\|_{S_R} \geq e^{-n_s \rho} e^{n_s \epsilon}.$$

Taking logarithms this may be rewritten as:

$$(2.13) \quad \left\| -R(t) + \frac{1}{n_s} \sum_{j=1}^{n_s} \log |t - z_j^s| \right\|_{S_R} \geq -\rho + \epsilon.$$

Now for $t \in S_R$, it follows from 2.11 that

$$(2.14) \quad \limsup_s \frac{1}{n_s} \sum_{j=1}^{n_s} \log |t - z_j^s| \leq \int \log |t - \xi| d\mu_{K,R}(\xi) = R(t) - \rho$$

where the equality is due to 2.6.

By Hartogs' lemma, ([13], Theorem 2.6.4) we have

$$(2.15) \quad \limsup_s \frac{1}{n_s} \sum_{j=1}^{n_s} \log |t - z_j^s| \leq R(t) - \rho + \epsilon/2,$$

uniformly on S_R .

That is, for s sufficiently large,

$$(2.16) \quad \left\| -R(t) + \frac{1}{n_s} \sum_{j=1}^{n_s} \log |t - z_j^s| \right\|_{S_R} \leq -\rho + \epsilon/2,$$

which contradicts 2.13. □

Corollary 2.2. *Under the hypothesis of Lemma 2.1*

$$(2.17) \quad \|e^{-2nQ(t)} |(t - z_1) \dots (t - z_n)|^\beta\|_K \leq e^{-n\beta(\rho - \epsilon)}.$$

Definition 2.3. Let τ be a positive Borel measure on a compact set $K \subset \mathbb{C}$ and R a real-valued continuous function on K . We say τ satisfies

the weighted Bernstein-Markov (BM) inequality for the weight e^{-R} if, for all $\epsilon > 0$, there exists $C > 0$ (independent of n) such that

$$(2.18) \quad \|e^{-nR}p\|_K \leq C(1+\epsilon)^n \int_K e^{-nR}|p|d\tau,$$

for all $p \in \mathcal{P}_n$.

It is known ([16], proof of theorem 3.4.3) that if τ satisfies 2.18 then it also satisfies an L^β version of that inequality (with, possibly, a different constant C), namely

$$(2.19) \quad \|e^{-nR}p\|_K \leq C(1+\epsilon)^n \left(\int_K (e^{-nR}|p|)^\beta d\tau \right)^{\frac{1}{\beta}}.$$

Remark 2.4. Let K_1 and K_2 be compact subsets of K with $S_R \subset K_1 \subset K_2$. It follows from 2.8 that if τ satisfies the weighted BM inequality for e^{-R} on K_1 then it satisfies the weighted BM inequality for e^{-R} on K_2 .

Combining 2.19 with 2.10 and 2.8 we have for p monic of degree n .

$$(2.20) \quad \int_K |e^{-2nQ}p^\beta|d\tau \geq C(1+\epsilon)^{-n\beta} \|e^{-2nQ}p^\beta\|_K = C(1+\epsilon)^{-n\beta} \|e^{-nR}p\|_{S_R}^\beta \geq C(1+\epsilon)^{-n\beta} e^{-n\rho\beta}.$$

We note that 2.20, for appropriate $C > 0$ is, in fact, valid for monic polynomials p of degree n or $n-1$.

3. JOHANSSON LARGE DEVIATION

We will not use the l.d.p. for the normalized counting measure of a random point but a weaker result whose utility was shown by Johansson [12].

Consider

$$(3.1) \quad A_{n,\beta,Q}(z) := |VDM(z_1, \dots, z_n)|^\beta \exp(-2n[Q(z_1) + \dots + Q(z_n)])$$

where $\beta > 0$, VDM denotes the VanDerMonde determinant

$$VDM(z_1, \dots, z_n) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$$

and Q is a continuous, real-valued function on K . Let τ satisfy the weighted BM inequality on K for e^{-R} . Let

$$(3.2) \quad Z_{n,\beta,Q} := \int_{K^n} A_{n,\beta,Q}(z) d\tau(z)$$

where

$$d\tau(z) = d\tau(z_1) \dots d\tau(z_n).$$

We define a probability measure on K^n for $n = 1, 2, \dots$ by

$$(3.3) \quad \text{Prob}_{n,\beta,Q} = \frac{A_{n,\beta,Q}(z)}{Z_{n,\beta,Q}} d\tau(z)$$

We obtain a collection of probability measures which we refer to as a β ensemble.

We let, for $\nu \in \mathcal{M}(K)$

$$(3.4) \quad E_\beta(\nu) = -\beta/2 \int \int \log|x-y| d\nu(x) d\nu(y) + 2 \int Q(x) d\nu(x).$$

Then

$$(3.5) \quad E_\beta(\nu) = \frac{\beta}{2} E(\nu).$$

Thus the unique minimizer in $\mathcal{M}(K)$ of $E_\beta(\nu)$ is $\mu_{K,R}$ for which we will also use the notation $\mu_{K,\beta,Q}$.

Let $F(n) = (f_1, \dots, f_n)$ be a point in K^n at which $A_{n,\beta,Q}(z)$ assumes its maximum. Then that point is also a point at which

$$(3.6) \quad A_{n,\beta,Q}^{2/\beta}(z) = |VDM(z_1, \dots, z_n)|^2 \exp(-2n[R(z_1) + \dots + R(z_n)])$$

assumes its maximum. It is known that ([5], prop 4.1 or [15], III, remark 1.4) we have

$$(3.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \delta(f_j) = \mu_{K,\beta,Q} = \mu_{K,R}$$

weak*.

Theorem 3.1.

$$\lim_n \frac{1}{n^2} \log Z_{n,\beta,Q} = -E_\beta(\mu_{K,\beta,Q}) = \lim_n \frac{1}{n^2} \log A_{n,\beta,Q}(F(n))$$

Proof. $\log A_{n,\beta,Q}(z)$ is a discrete approximation to $-E_\beta$. The proof is then analogous to the proofs in section 3 of [2] (see also [6]). \square

Let $\log \gamma := -E_\beta(\mu_{K,\beta,Q})$. Then $\gamma > 0$ and let $\eta > 0, \eta < \gamma$ be given. We define

$$B_{\eta,n,\beta}^Q := \{z \in K^n | A_{n,\beta,Q}^{\frac{1}{n^2}}(z) \leq \gamma - \eta\}.$$

Then we have the following result, which we refer to as a Johansson large deviation result:

Theorem 3.2.

$$Prob_{n,\beta,Q}(B_{\eta,n,\beta}^Q) = \frac{1}{Z_{n,\beta,Q}} \int_{B_{\eta,n,\beta}^Q} A_{n,\beta,Q}(z) d\tau(z) \leq (1 - \frac{\eta}{2\gamma})^{n^2}$$

for all n sufficiently large.

Proof. Since, by theorem 3.1,

$$\lim_n \frac{1}{n^2} \log \sup_{K^n} A_{n,\beta,Q}(z) = \lim_n \frac{1}{n^2} \int_{K^n} A_{n,\beta,Q}(z) d\tau(z) = \log \gamma > -\infty$$

the considerations of [2], section 4 apply (see also [6]) and the result follows. \square

Theorem 3.3. *Let G be a neighbourhood of $\mu_{K,\beta,Q}$ in $\mathcal{M}(K)$. Then*

$$\frac{1}{Z_{n,\beta,Q}} \int_{K^n \setminus \tilde{G}_n} A_{n,\beta,Q}(z) d\tau \leq O(e^{-cn^2})$$

for some $c > 0$.

Proof. It follows from the reasoning used in proposition 7.3 of [2] (see also [6]) that for some $\eta > 0$,

$$B_{\eta,n,\beta}^Q \supset K^n \setminus \tilde{G}_n$$

for all n sufficiently large, and so the result then follows from Theorem 3.2. \square

4. THE NORMALIZING CONSTANTS

Theorem 3.1 gives an asymptotic result for the normalizing constants $Z_{n,\beta,Q}$. Note that if we have a sequence of continuous functions $\{Q_n\}$ converging uniformly to Q on K , then $\frac{1}{n^2} \log Z_{n,\beta,Q}$ and $\frac{1}{n^2} \log Z_{n,\beta,Q_n}$ have the same limit, and in particular this is true for $\frac{1}{n^2} \log Z_{n,\beta,\frac{nQ}{n-1}}$. We will, however, need a sharper result, namely:

Theorem 4.1.

$$\lim_{n \rightarrow \infty} \left(\frac{Z_{n,\beta,Q}}{Z_{n-1,\beta,\frac{nQ}{n-1}}} \right)^{\frac{1}{n}} = e^{-\rho\beta}.$$

Proof. We will first prove that

$$(4.1) \quad \liminf_{n \rightarrow \infty} \left(\frac{Z_{n,\beta,Q}}{Z_{n-1,\beta,\frac{nQ}{n-1}}} \right)^{\frac{1}{n}} \geq e^{-\rho\beta}.$$

Now $Z_{n,\beta,Q} = \int_{K^n} A_{n,\beta,Q}(z) d\tau(z)$. We regard $A_{n,\beta,Q}(z)$ as a function of z_1 and we apply 2.20 to the integral in the z_1 variable to obtain

$$(4.2) \quad \begin{aligned} Z_{n,\beta,Q} &\geq C(1+\epsilon)^{-n\beta} e^{-n\rho\beta} \int_{K^n} |VDM(z_2, \dots, z_n)|^\beta e^{-2n[Q(z_2)+\dots+Q(z_n)]} d\tau(z_2) \dots d\tau(z_n) \\ &= C(1+\epsilon)^{-n\beta} e^{-n\rho\beta} Z_{n-1,\beta,\frac{nQ}{n-1}} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the estimate on the lower limit follows.

To complete the proof we must show

$$(4.3) \quad \limsup_{n \rightarrow \infty} \left(\frac{Z_{n,\beta,Q}}{Z_{n-1,\beta,\frac{nQ}{n-1}}} \right)^{\frac{1}{n}} \leq e^{-\rho\beta}.$$

We write the integral for $Z_{n,\beta,Q}$ as a sum of two integrals

$$Z_{n,\beta,Q} = I_1 + I_2$$

where

$$I_1 = \int_K \int_{K^{n-1} \setminus \tilde{G}_{n-1}} A_{n,\beta,Q}(z) d\tau(z)$$

and

$$I_2 = \int_K \int_{\tilde{G}_{n-1}} A_{n,\beta,Q}(z) d\tau(z).$$

\tilde{G}_{n-1} is a subset of K^{n-1} determined as follows: Given $\epsilon > 0$ choose a neighborhood G of $\mu_{K,\beta,Q}$ so that Corollary 2.2 holds. Then

$$\tilde{G}_{n-1} = \{(z_2, \dots, z_n) \in K^{n-1} \mid \frac{1}{n-1} \sum_{j=2}^n \delta(z_j) \in G\}.$$

Now,

$$(4.4) \quad \begin{aligned} I_1 &= \int_K \prod_{j=2}^n |z_1 - z_j|^\beta e^{-2nQ(z_1)} d\tau(z_1) \times \\ &\quad \int_{K^{n-1} \setminus \tilde{G}_{n-1}} |VDM(z_2, \dots, z_n)|^\beta e^{-2n[Q(z_2)+\dots+Q(z_n)]} d\tau(z_2) \dots d\tau(z_n). \end{aligned}$$

The first factor is $O(C^n)$ for some $C > 0$. The integrand in the second factor differs from $A_{n-1,\beta,Q}(z_2, \dots, z_n)$ by a factor of $O(C^n)$ for some

$C > 0$ so the second factor is $O(e^{-cn^2})Z_{n-1,\beta,Q}$ by Theorem 3.3. Since

$$Z_{n-1,\beta,Q} = O(C^n)Z_{n-1,\beta,\frac{nQ}{n-1}},$$

we may conclude that

$$I_1 = O(e^{-cn^2})Z_{n-1,\beta,\frac{nQ}{n-1}}.$$

Also

$$(4.5) \quad I_2 = \int_K \prod_{j=2}^n |z_1 - z_j|^\beta e^{-2nQ(z_1)} d\tau(z_1) \times \\ \int_{\tilde{G}_{n-1}} |VDM(z_2, \dots, z_n)|^\beta e^{-2n[Q(z_2) + \dots + Q(z_n)]} d\tau(z_2) \dots d\tau(z_n).$$

We will need a more precise estimate on the first factor than the one used in 4.4. Since we are integrating over \tilde{G}_{n-1} we may use 2.17 on the first factor to see that it is $\leq Ce^{-n\beta(\rho-\epsilon)}$ and the second factor is $\leq Z_{n-1,\beta,\frac{nQ}{n-1}}$ since \tilde{G}_{n-1} is a subset of K^{n-1} . Hence, given any $\epsilon > 0$, we have for some $c > 0$

$$I_1 + I_2 \leq Ce^{-n\beta(\rho-\epsilon)}Z_{n-1,\beta,\frac{nQ}{n-1}} + O(e^{-cn^2})Z_{n-1,\beta,\frac{nQ}{n-1}}$$

and 4.3 follows. \square

5. LARGE DEVIATION

Recall that given a separable, complete metric space X , a sequence of probability measures $\{\sigma_n\}$ on X is said to satisfy a large deviation principle with speed n and rate function $\mathbb{J}(x)$ if $\mathbb{J} : X \rightarrow [0, \infty]$ is lower-semicontinuous, $\{x \in X | \mathbb{J}(x) \leq l\}$ is compact for $l \geq 0$ and

(i) For all closed sets $F \subset X$ we have

$$\limsup_n \frac{1}{n} \log \sigma_n(F) \leq - \inf_{x \in F} \mathbb{J}(x)$$

(ii) For all open sets $G \subset X$ we have

$$\liminf_n \frac{1}{n} \log \sigma_n(G) \geq - \inf_{x \in G} \mathbb{J}(x)$$

If X is compact by ([8], theorem 4.1.11) to establish the l.d.p. it suffices to show that, for all $x \in X$

$$(5.1) \quad -\mathbb{J}(x) = \lim_{\epsilon \rightarrow 0} \lim_n \frac{1}{n} \log \sigma_n(B(x, \epsilon))$$

where $B(x, \epsilon)$ is the ball center x , radius ϵ .

If X is non-compact there is an additional condition required, termed *exponential tightness*, namely:

For all $r > 0$ there is a compact set $X_r \subset X$ with $\sigma_n(X_r) \leq -r$.

We will prove a large deviation principle for z_1 on a compact set K and in section 6 we will extend the l.d.p. to the case of non-compact sets.

That is we will prove a large deviation principle for the countable family of probability distributions on K given by

$$(5.2) \quad \psi_n(z_1) = \left(\frac{1}{Z_{n,\beta,Q}} \int_{K^{n-1}} A_{n,\beta,Q}(z) d\tau(z_2), \dots, d\tau(z_n) \right) d\tau(z_1).$$

for $n = 1, 2, \dots$

Recall that $R = 2Q/\beta$ and that $V_{Y,R}$ (see 2.7) is the weighted Green function of Y with respect to R .

We will assume that 2.5 holds. We will also assume that the measure τ satisfies the weighted BM inequality for e^{-R} on any compact neighbourhood S_R .

Theorem 5.1. *Let K be a regular, compact subset of \mathbb{C} , and Q a real-valued continuous function on K . Then z_1 satisfies a l.d.p. on K with speed n and rate function*

$$\mathbb{J}_{K,\beta,Q}(z_1) = \beta(R(z_1) - V_{K,R}(z_1)) = 2Q(z_1) - \beta V_{K,R}(z_1).$$

Proof. Using 2.3 we have

$$\mathbb{J}_{K,\beta,Q}(z_1) = 2Q(z_1) - \beta \left(\int_K \log |z_1 - t| d\mu_{K,\beta,Q}(t) + \rho \right)$$

Note that $\mathbb{J}_{K,\beta,Q}(z_1) = 0$ for $z_1 \in S_R$.

We will use 5.1. First we consider the case $z_1 \notin S_R$. Let W be a neighbourhood of z_1 with $\overline{W} \cap S_R = \emptyset$.

We will estimate the probability that a point $w \in W$.

By definition,

$$(5.3) \quad \text{Prob}_{n,\beta,Q}\{w \in W\} = \frac{1}{Z_{n,\beta,Q}} \int_W \int_{K^{n-1}} A_{n,\beta,Q}(z) d\tau(z).$$

We will separately estimate

$$\limsup_n 1/n \log(\text{Prob}_{n,\beta,Q}\{w \in W\}) \text{ and } \liminf_n 1/n \log(\text{Prob}_{n,\beta,Q}\{w \in W\}).$$

We begin with the \limsup .

We write the above integral as a sum of two integrals (where G is an open neighbourhood of $\mu_{K,\beta,Q} \subset \mathcal{M}(K)$ which is to be specified).

$$\frac{1}{Z_{n,\beta,Q}} \int_W \int_{K^{n-1}} A_{n,\beta,Q}(z) d\tau(z) = H_1 + H_2.$$

$$H_1 = \frac{1}{Z_{n,\beta,Q}} \int_W \int_{K^{n-1} \setminus \tilde{G}_{n-1}} A_{n,\beta,Q}(z) d\tau(z)$$

and

$$H_2 = \frac{1}{Z_{n,\beta,Q}} \int_W \int_{\tilde{G}_{n-1}} A_{n,\beta,Q}(z) d\tau(z).$$

Similar to the estimate for I_1 , (see 4.4) we have $H_1 = O(e^{-cn^2})$.

To estimate H_2 we now proceed as in [1]. Let

$$(5.4) \quad h_n(K) = h_n := \frac{Z_{n,\beta,Q}}{Z_{n-1,\beta,\frac{nQ}{n-1}}} = \frac{Z_{n,\beta,Q}(K)}{Z_{n-1,\beta,\frac{nQ}{n-1}}(K)}.$$

Then

$$(5.5) \quad H_2 = \frac{1}{h_n Z_{n-1,\beta,\frac{nQ}{n-1}}} \int_W \int_{\tilde{G}_{n-1}} A_{n,\beta,Q}(z) d\tau(z).$$

Now, given $\epsilon > 0$, let G be a neighborhood of $\mu_{K,\beta,Q} \subset \mathcal{M}(K)$ so that for $w \in W$ and $(z_2, \dots, z_n) \in \tilde{G}_{n-1}$ then

$$(5.6) \quad \beta \left(\frac{1}{n-1} \sum_{j=2}^n \log |w - z_j| - \int \log |w - t| d\mu_{K,\beta,Q}(t) \right) \leq \epsilon.$$

For the existence of such a neighborhood G see the reasoning used in lemma 2.1.

Taking exponentials

$$(5.7) \quad \prod_{j=2}^n |w - z_j|^\beta \leq e^{(n-1)(\epsilon + \beta \int \log |w-t| d\mu_{K,\beta,Q}(t))}.$$

Also, since $\tilde{G}_{n-1} \subset K^{n-1}$

$$(5.8) \quad \frac{1}{Z_{n-1,\beta,\frac{nQ}{n-1}}} \int_{\tilde{G}_{n-1}} |VDM(z_2, \dots, z_n)|^\beta e^{-2n[Q(z_2)+\dots+Q(z_n)]} d\tau(z_2) \dots d\tau(z_n) \leq 1.$$

Using theorem 4.1 to estimate h_n , and the inequalities in 5.7 and 5.8, we have

$$(5.9) \quad \limsup_n 1/n \log(Prob_{n,\beta,Q}\{w \in W\}) \leq \beta\rho + \sup_{w \in W} (\beta \int_K \log|w-t| d\mu_{K,\beta,Q} - 2Q(w)).$$

Now we must deal with the \liminf . We begin with the lemma:

Lemma 5.2. *Let $N \subset K$ be a compact neighbourhood of S_R . There is a constant $c > 0$ (independent of n, p) such that:*

$$\int_K |e^{-nR}p|^\beta d\tau \leq (1 + O(e^{-nc})) \int_N |e^{-nR}p|^\beta d\tau$$

for all $p \in \mathcal{P}_n$.

Proof. We first normalize the polynomial so that $\|e^{-nR}p\|_{S_R} = 1$. To prove the theorem it will suffice to show that

$$(5.10) \quad \int_{K \setminus N} |e^{-nR(z)}p(z)|^\beta d\tau \leq Ce^{-nc},$$

for some constants $C, c > 0$ and

$$(5.11) \quad \liminf_n \left(\int_K |e^{-nR}p|^\beta d\tau \right)^{1/n} \geq 1.$$

We will use the estimate ([15], appendix B, theorem 2.6 (ii))

$$(5.12) \quad |e^{-nR(z)}p(z)| \leq \|e^{-nR(z)}p(z)\|_{S_R} \exp(n(V_{K,R}(z) - R(z)))$$

for $z \in K$.

For $z \in K \setminus N$, and some constant $b > 0$

$$(5.13) \quad V_{K,R}(z) - R(z) \leq -b < 0$$

so $|e^{-nR(z)}p(z)| \leq e^{-nb}$ for $z \in K \setminus N$

Thus,

$$(5.14) \quad \int_{K \setminus N} |e^{-nR(z)}p(z)|^\beta d\tau \leq Ce^{-nb\beta}$$

for constants $C, b > 0$.

Now,

$$(5.15) \quad \int_K |e^{-nR(z)} p(z)|^\beta d\tau \geq \int_N |e^{-nR(z)} p(z)|^\beta d\tau$$

and since τ satisfies the weighted BM condition on N for e^{-R} the right hand side in 5.15 is, for any $\epsilon > 0$ and some $C > 0$

$$(5.16) \quad \geq \|e^{-nR} p\|_{S_R}^\beta C e^{-\epsilon n} = C e^{-\epsilon n},$$

establishing 5.24 and the result. \square

Corollary 5.3.

$$\int_K |e^{-2nQ} p^\beta| d\tau \leq (1 + O(e^{-nc})) \int_N |e^{-2nQ} p^\beta| d\tau$$

for all $p \in \mathcal{P}_n$.

Now $A_{n,\beta,Q}(z)$ is in each variable of the form $e^{-2nQ}|p|^\beta$ for a polynomial $p \in \mathcal{P}_n$. Hence by repeated use of corollary 5.3 we have, (see also [7], section 2.2)

$$(5.17) \quad \int_{K^n} A_{n,\beta,Q}(z) d\tau(z) \leq (1 + O(e^{-cn})) \int_{N^n} A_{n,\beta,Q}(z) d\tau(z),$$

or

$$(5.18) \quad Z_{n,\beta,Q}(K) \leq (1 + O(e^{-cn})) Z_{n,\beta,Q}(N)$$

Similarly,

$$(5.19) \quad Z_{n-1,\beta,\frac{nQ}{n-1}}(K) \leq (1 + O(e^{-cn})) Z_{n-1,\beta,\frac{nQ}{n-1}}(N)$$

Now let N be a compact neighbourhood of S_R such that $N \cap \overline{W} = \emptyset$

$$\frac{1}{Z_{n,\beta,Q}(K)} \int_W \int_{K^{n-1}} A_{n,\beta,Q}(z) d\tau(z) \geq \frac{1}{Z_{n,\beta,Q}(K)} \int_W \int_{N^{n-1}} A_{n,\beta,Q}(z) d\tau(z)$$

and using 5.17, to obtain the lower bound it suffices to estimate:

$$\frac{1}{Z_{n,\beta,Q}(N)} \int_W \int_{N^{n-1}} A_{n,\beta,Q}(z) d\tau(z) = \frac{1}{h_n(N) Z_{n-1,\beta,\frac{nQ}{n-1}}(N)} \int_W \int_{N^{n-1}} A_{n,\beta,Q}(z) d\tau(z)$$

Given $\epsilon > 0$ let F be a neighbourhood of $\mu_{K,\beta,Q}$ in $\mathcal{M}(N)$ such that for $w \in W$ and $(z_2, \dots, z_n) \in \tilde{F}_{n-1}$ we have

$$(5.20) \quad -\epsilon \leq \beta \left(\frac{1}{n-1} \sum_{j=2}^n \log |w - z_j| - \int \log |w - t| d\mu_{K,\beta,Q}(t) \right).$$

or, taking exponentials

$$(5.21) \quad \prod_{j=2}^n |w - z_j|^\beta \geq e^{(n-1)(-\epsilon + \beta \int \log |w - t| d\mu_{K,\beta,Q}(t))}.$$

Now for $(z_2, \dots, z_n) \in \tilde{F}_{n-1}$ we have (see 4.4)

$$(5.22) \quad \frac{1}{Z_{n-1,\beta,\frac{nQ}{n-1}}(N)} \int_{N^{n-1} \setminus \tilde{F}_{n-1}} |VDM(z_2, \dots, z_n)|^\beta e^{-2n[Q(z_2) + \dots + Q(z_n)]} d\tau(z_2) \dots d\tau(z_n) \leq O(e^{-cn^2}).$$

So

$$(5.23) \quad 1 - O(e^{-cn^2}) \leq \frac{1}{Z_{n-1,\beta,\frac{nQ}{n-1}}(N)} \int_{\tilde{F}_{n-1}} |VDM(z_2, \dots, z_n)|^\beta e^{-2n[Q(z_2) + \dots + Q(z_n)]} d\tau(z_2) \dots d\tau(z_n)$$

Using the inequalities in 5.21 and 5.23 we have

$$(5.24) \quad \liminf_n \frac{1}{n} \log(Prob_{n,\beta,Q}\{w \in W\}) \geq \beta \rho + \inf_{w \in W} \left(\beta \int_K \log |w - t| d\mu_{K,\beta,Q} - 2Q(w) \right).$$

Since $w \rightarrow \int_K \log |w - t| d\mu_{K,\beta,Q}$ is continuous for $w \in W$ it follows that

$$(5.25) \quad \inf_{W \ni z_1} \lim_n \frac{1}{n} \log Prob_{n,\beta,Q}\{w \in W\} = \beta \left(\rho + \int_K \log |z_1 - t| d\mu_{K,\beta,Q} - 2Q(z_1) \right).$$

To complete the l.d.p. we must consider the case when $z_1 \in S_R$ and to do so we must estimate $Prob_{n,\beta,Q}\{w \in W\}$ when $W \cap S_R \neq \emptyset$. In fact, we will show that, in this case,

$$(5.26) \quad \lim_n \frac{1}{n} \log Prob_{n,\beta,Q}\{w \in W\} = 0.$$

Now

$$\begin{aligned} n Prob_{n,\beta,Q}\{w \in W\} &\geq Prob_{n,\beta,Q}\{\text{at least one of } w, z_2, \dots, z_n \in W\} \\ &= 1 - Prob_{n,\beta,Q}\{\text{each of } w, z_2, \dots, z_n \in K \setminus W\} \end{aligned}$$

Since $W \cap S_R \neq \emptyset$ the support of the weighted equilibrium measure for R on $K \setminus W$ cannot be S_R . This means that, using theorem 3.1 and the minimizing property of the equilibrium measure, that $\limsup_n \sup_{K \setminus W} A_{n,\beta}(z)^{\frac{1}{n^2}} \leq \gamma - \eta$, for some $\eta > 0$. Then one can use theorem 3.2 to obtain

$$(5.27) \quad \text{Prob}_{n,\beta,Q}\{w \in W\} \geq \frac{1}{n}(1 - O(e^{-cn^2}))$$

and 5.26 follows. \square

6. THE UNBOUNDED CASE

Let Y be a closed, unbounded and regular subset of \mathbb{C} . Let R be a continuous, super-logarithmic function on Y . That is, for some $b > 0$

$$(6.1) \quad \lim_{|z| \rightarrow \infty} R(z) - (1+b) \log |z| = +\infty.$$

For $r > 0$ we let $Y_r =: \{z \in Y \mid |z| \leq r\}$. Then for r sufficiently large $V_{Y,R} = V_{Y_r,R}$ ([15], Appendix B, Lemma 2.2) and $S_R \subset Y_r$. We will also denote the equilibrium measure $\mu_{Y,R}$ by $\mu_{Y,\beta,Q}$. We will assume 2.5 holds.

Let τ be a locally finite positive Borel measure on Y satisfying:

$$(6.2) \quad \text{For some } a > 0, \text{ we have } \int_Y d\tau / |z|^a < +\infty$$

$$(6.3)$$

τ satisfies the weighted BM inequality for e^{-R} on sufficiently small compact neighbourhoods of S_R .

We note that Lebesgue measure on \mathbb{R} or \mathbb{C} satisfies the above since Lebesgue measure satisfies the weighted BM inequality for any weight on intervals of \mathbb{R} or smoothly bounded subsets of \mathbb{R}^2

Under the above assumptions we will extend lemma 5.2 to the unbounded case.

The following theorem is based on ([15], III Theorem 6.1). It shows that the L^β norm of a weighted polynomial "lives" on S_R .

Theorem 6.1. *Given $\beta > 0$ and $N \subset Y$ a compact neighbourhood of S_R , then there is a constant $c > 0$ (independent of n, p) such that:*

$$\int_Y |e^{-nR} p|^\beta d\tau \leq (1 + O(e^{-nc})) \int_N |e^{-nR} p|^\beta d\tau$$

for all $p \in \mathcal{P}_n$.

Proof. Using lemma 5.2 it suffices to deal with the case $N = Y_r$ and we proceed as in lemma 5.2. We first normalize the polynomial so that $\|e^{-nR}p\|_{S_R} = 1$. To prove the theorem it will suffice to show that

$$(6.4) \quad \int_{Y \setminus Y_r} |e^{-nR(z)}p(z)|^\beta d\tau \leq Ce^{-nc},$$

for some constants $C, c > 0$ and

$$(6.5) \quad \liminf_n \left(\int_Y |e^{-nR}p|^\beta d\tau \right)^{1/n} \geq 1.$$

We will use the estimate ([15], appendix B, theorem 2.6 (ii))

$$(6.6) \quad |e^{-nR(z)}p(z)| \leq \|e^{-nR(z)}p(z)\|_{S_R} \exp(n(V_{Y,R}(z) - R(z)))$$

for $z \in Y$.

Using 6.1 we have, since $V_{Y,R} \leq \log |z| + C$ for $|z|$ large

$$(6.7) \quad V_{Y,R}(z) - R(z) \leq -\frac{b}{2} \log |z|$$

for $|z|$ large .

Hence, for r large

$$(6.8) \quad \begin{aligned} \int_{Y \setminus Y_r} |e^{-nR(z)}p(z)|^\beta d\tau &\leq C \int_{Y \setminus Y_r} \frac{d\tau}{|z|^{\frac{nb\beta}{2}}} \\ &\leq Cr^{-\frac{nb\beta}{2}} r^a \int_{Y \setminus Y_r} \frac{d\tau}{|z|^a} \leq Ce^{-nc}. \end{aligned}$$

for constants $C, c > 0$ and all n sufficiently large.

Now,

$$(6.9) \quad \int_Y |e^{-nR(z)}p(z)|^\beta d\tau \geq \int_{Y_r} |e^{-nR(z)}p(z)|^\beta d\tau$$

and 6.5 follows from the proof of lemma 5.2. □

Corollary 6.2.

$$\int_Y |e^{-2nQ}p^\beta| d\tau \leq (1 + O(e^{-nc})) \int_N |e^{-2nQ}p^\beta| d\tau$$

for all $p \in \mathcal{P}_n$.

We have the following l.d.p. for z_1 on Y (under that assumptions at the beginning of section 6)

Theorem 6.3. z_1 satisfies a l.d.p. on Y with speed n and rate function $\mathbb{J}_{Y,\beta,Q} = \beta(R(z_1) - V_{Y,R}(z_1)) = 2Q(z_1) - \beta(\int_Y \log |z_1 - t| d\mu_{Y,\beta,Q}(t) + \rho)$.

Proof. We will show that 5.1 holds, together with exponential tightness.

We consider an open set $W \subset Y$. We may assume that $W \subset Y_r$. Applying the results of section 5 to β ensembles on the compact set Y_r for r sufficiently large, we have

$$(6.10) \quad \inf_{W \ni z_1} \lim_n \frac{1}{n} \log \text{Prob}_{n,\beta,Q}\{z_1 \in W\} = \beta(V_{Y,R}(z_1) - R(z_1)).$$

We will show the same result holds for β ensembles on Y .

Now $A_{n,\beta,Q}(z)$ and $\int_W A_{n,\beta,Q}(z) d\tau(z_1)$, are, in each variable of the form $e^{-2nQ}|p|^\beta$ for a polynomial $p \in \mathcal{P}_n$. Hence by repeated use of corollary 6.2 we have, (see also [7], section 2.2)

$$(6.11) \quad \int_{Y^n} A_{n,\beta,Q}(z) d\tau(z) \leq (1 + O(e^{-cn})) \int_{N^n} A_{n,\beta,Q}(z) d\tau(z),$$

and

$$(6.12) \quad \int_W \int_{Y^{n-1}} A_{n,\beta,Q}(z) d\tau(z) \leq (1 + O(e^{-cn})) \int_W \int_{N^{n-1}} A_{n,\beta,Q}(z) d\tau(z).$$

Note that 6.11 shows that the integrals defining the normalizing constants 1.3 are finite.

It also follows that taking $N = Y_r$ whether we consider β ensembles on Y or Y_r that $\lim_n \frac{1}{n} \log \text{Prob}_{n,\beta,Q}\{z_1 \in W\}$ will be the same.

To complete the large deviation property in the unbounded case we must establish exponential tightness. That is :

$$(6.13) \quad \lim_n \frac{1}{n} \log \text{Prob}_{n,\beta,Q}\{|z_1| > r\} \rightarrow -\infty \text{ as } r \rightarrow \infty$$

Proceeding as in the proof of 6.8 but without normalizing the polynomial, we obtain

$$(6.14) \quad \begin{aligned} \int_{Y \setminus Y_r} |e^{-nR(z)} p(z)|^\beta d\tau &\leq \|e^{-nR(z)} p(z)\|_{S_R}^\beta \int_{Y \setminus Y_r} \frac{d\tau}{|z|^{\frac{nb\beta}{2}}} \\ &\leq C \|e^{-nR(z)} p(z)\|_{S_R}^\beta r^{-\frac{nb\beta}{2} + a}. \end{aligned}$$

where C is independent of n, p .

By the weighted BM inequality, we have

$$(6.15) \quad \begin{aligned} \|e^{-nR(z)}p(z)\|_{S_R}^\beta &\leq C(1+\epsilon)^n \int_{Y_r} |e^{-nR}p|^\beta d\tau \\ &\leq C(1+\epsilon)^n \int_Y |e^{-nR}p|^\beta d\tau. \end{aligned}$$

Now $\psi_n(z_1)$ (see 1.4) is an integral of functions of the form $|e^{-nR}p|^\beta$ depending on z_2, \dots, z_n so using Fubini's theorem and 6.14, 6.15, we have

$$(6.16) \quad \int_{Y \setminus Y_r} \psi_n(z_1) d\tau(z_1) \leq C(1+\epsilon)^n r^{\frac{-nb\beta}{2}+a} \int_Y \psi_n(z_1) d\tau(z_1)$$

Since $\epsilon > 0$ is arbitrary and ψ_n is a probability distribution 6.13 and hence Theorem 6.3 is established. \square

Remark 6.4. In ([1], theorem 2.6.6) a l.d.p for the largest eigenvalue in β ensembles on the real line is established. The rate function is the same as the one given in theorem 6.3 above for points $x > x^*$ where $x^* = \max\{x \in S_R\}$. This is because, given an open interval (a, b) with $b > x^*$ we take $W = (a, b)$ and $U = (-\infty, b)$ and $x^M = \max\{x_1, \dots, x_n\}$, then

$$(6.17) \quad \begin{aligned} n \text{Prob}_{n,\beta,Q}\{x_1 \in W, x_2, \dots, x_n \in U\} &\geq \text{Prob}_{n,\beta,Q}\{x^M \in W\} \\ &\geq \text{Prob}_{n,\beta,Q}\{x_1 \in W, x_2, \dots, x_n \in U\}. \end{aligned}$$

Then using 6.12 it follows that

$$(6.18) \quad \lim_n \frac{1}{n} \log \text{Prob}_{n,\beta,Q}\{x^M \in W\} = \lim_n \frac{1}{n} \log \text{Prob}_{n,\beta,Q}\{x_1 \in W\}.$$

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